KALMAN FILTER MODELS FOR EXTRAPOLATIONS IN DOSE-RESPONSE EXPERIMENTS AND ACCELERATED LIFE-TESTS

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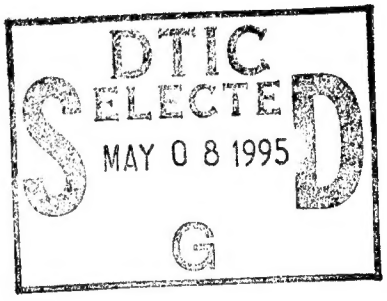

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Abstract

Kalman-Filter models with Gaussian innovations provide a useful and easy to implement tool for inference from dose-response experiments and accelerated life-tests. Their main advantage stems from the fact that the system equation of such models alllows for the uncertainty and possible changes in a proposed dose-response relationship. This is in contrast to the currently used approaches wherein there is an implicit commitment to the validity of an assumed relationship. In this paper, we overview our recent work in the above general area and suggest avenues for future research.

1. INTRODUCTION

In its simplest form, a Kalman-Filter model relates an observed value Y_t at time t, t=1, 2,..., to an unknown (vector or scalar) quantity θ_t – the state of nature – via the observation equation $Y_t = F_t \theta_t + r_t$, where F_t is a known coefficient and (r_t) a sequence of innovations assumed to be Gaussian with known parameters. What distinguishes a Kalman-Filter model from the standard regression models, is the consideration of a system equation wherein θ_t is related to its previous value θ_{t-1} via the relationship $\theta_t = G_t\theta_{t-1} + w_t$, where G_t is a known coefficient and (w_t) a sequence of uncorrelated innovations, assumed Gaussian with known parameters. Furthermore, the w_t's are contemporaneously uncorrelated with the r_t's. Given the observations $Y_1, Y_2,...,Y_t$, the Kalman-Filter algorithm enables us to obtain the posterior distribution of θ_t in a recursive manner. The algorithm is started-off by specifying the mean and the covariance of the prior distribution of θ_o. Literature on Kalman-Filter models and algorithms for the efficient computation of the posterior distribution of θ_t, under various scenarios and generalizations of the above theme, is typically available in the engineering and control theory journals. Statisticians have recently become interested in such models, and a possible source of introduction is the expository paper by Meinhold and Singpurwalla (1983).

The accelerated life-testing set-up can be best described via the following scenario: we are given stresses (or doses) $S_1 > S_2 \ldots > S_t > S_{t+1}$, and under S_j an item has life-length T_j, with the T_j's assumed stochastically increasing in j. Given the observed realizations of $T_1, T_2,...,T_t$, we need to make statements of uncertainty about T_{t+1}; we may or may not have realizations of T_{t+1}. It is common – though rather restrictive – to assume that T_j has an exponential distribution

with a mean θ_j, and so what is desired is inference about θ_{t+1}. An important ingredient for inference about θ_{t+1} given the realizations of $T_1,...,T_t$ (and perhaps T_{t+1}), is a relationship (or model) which connects the θ's and the S's. Such a relationship is referred to as a time transformation function, and the choice of such functions should be dictated by considerations embedded in the physics, the chemistry or the biology of the failure generating mechanism. This is an issue which calls for collaboration between statisticians and their colleagues in the engineering and the natural sciences and presents an opportunity for meaningful new work in accelerated life testing. The current literature on accelerated life testing focuses attention on two commonly used time transformation functions, both of which appear to have a naïve physical basis. These are, the Power Law, under which $\theta_j = \alpha_j S_j^{\phi_j}$, and the Arrhenius' Law, under which $\theta_j = \exp(\alpha_j + \phi_j/S_j)$, where α_j and ϕ_j are unknown constants – see Mann, Schafer and Singpurwalla (1974), p. 421. Notwithstanding the comment that the above two relationships may have a naïve physical basis, there is also some concern that the validity of these relationships may not hold over the entire range of the stresses. High values of the stress may trigger a change in the basic failure generating mechanism calling for different time transformation functions over different ranges of the stress. Casting the time transformation function as the system equation of a Kalman-Filter model, enables us - via the innovation terms w_t – not only to incorporate a measure of uncertainty about the time transformation function, but also to better track changes in such functions.

The dose-response and the damage assessment scenarios that are of interest to us here, can be described as follows: we are given a dose (or stress) say X, with X taking values x_j, j=1,...,T, and $x_j \in [0, \infty]$. We may or may not be able to

control the values x_j precisely, and under each x_j we are able to test one or more specimens of an item of interest. Let $Y(x_j)$ denote the response under dose x_j and let $Y(x_j) \in (0, 1)$; for example, $Y(x_j)$ could represent the proportion of items damaged or the extent of damage to a particular item. We assume that the response is a nonlinear function of the dose; specifically $E(Y(x)) = \exp(-\alpha(x)x^{\beta(x)})$, where $\alpha(x)$ and $\beta(x)$ are unknown constants. The above relationship, which resembles the survival function of a Weibull distribution, is like the time transformation function of accelerated life tests. It's key advantages are that it can be easily linearized and that it is flexible enough to represent many shapes that are candidates for describing the relationship between the dose and its response. Given the x_j's and their associated Y_j's, $j=1,...,T$, our aim is to make inferences about $\alpha(x)$ and $\beta(x)$ and use these to assess our uncertainty about $Y(x_o)$ for any $x_o \neq x_j$.

2. KALMAN-FILTER MODELS FOR ACCELERATED LIFE-TESTS

We have proposed two strategies for inference from accelerated life tests, using Kalman-Filter models. The first one, described in Meinhold and Singpurwalla (1984), is applicable when a large number of items are tested to failure at each stress level. Its main advantage is computational, since it results in a direct application of the Kalman-Filter algorithm with serially uncorrelated innovations for the observation equation. The second one, described in Blackwell and Singpurwalla (1988), is applicable for smaller sample sizes but requires a computationally more intensive effort, since the resulting observation equation of the Kalman-Filter contains serially correlated innovations. Engineers refer to Kalman-Filter models with correlated innovations in the system equation as filtering in coloured noise. We shall first outline the former of the above two strategies.

Strategy 1. Suppose that n_j items are tested to failure under stress S_j, and let X_j be the sample mean of the n_j observed life-lengths. Let $Y_j = -\log X_j$; then it can be shown that the cumulants of Y_j approach those of a Gaussian distribution, suggesting that the observation equation of a Kalman-Filter be formulated in terms of Y_j. Specifically, we let $Z_j = \log n_j - \psi(n_j) - Y_j$, where $\psi(n_j)$ is the digamma function, write the observation equation of the Kalman-Filter as $Z_j = \log \theta_j + r_j$, and argue that $r_j \sim \mathcal{N}(0, \psi'(n_j))$, where $\psi'(n_j)$ is the first derivative of the digamma function; the notation "$X \sim \mathcal{N}(\mu, \sigma)$" denotes the fact that X has a Gaussian distribution with mean μ and variance σ. The sequence of innovations $r_1, r_2, \ldots, r_t$, are assumed to be uncorrelated. The above form of the observation equation is motivated by the fact that K_1 and K_2, the first and the second cumulant of the distribution of Y_j, given θ_j, are of the form $K_1 = \log n_j - \log\theta_j - \psi(n_j)$ and $K_2 = \psi'(n_j)$.

For the system equations of the Kalman-Filter, we argue that the Arrhenius and the Power Law, suggest the general form

$$\log \theta_j = \log \theta_{j-1} + \phi_j \, h(S_j, S_{j-1}) + u_j, \text{ and}$$

$$\phi_j = \phi_{j-1} + v_j, \text{ with}$$

the vector $[u_j, v_j] \sim \mathcal{N}(0, \Sigma_j)$, where Σ_j is the covariance of u_j and v_j under stress j, $h(s_j, s_{j-1})$ is of the form $(S_j^{-1} - S_{j-1}^{-1})$ for the Arrhenius Law, with $u_j = v_j/S_{j-1}$, and of the form $\log(S_j/S_{j-1})$ with $u_j = v_j \log(S_{j-1}^{-1})$ for the Power Law. Since u_j and v_j, the components of the innovation term of the system equations are related to each other, we may write $u_j = \beta_j v_j + w_j$ is an innovation that is independent of v_j and $\beta_j = S_{j-1}^{-1} (\log S_{j-1}^{-1})$ for the Arrhenius (Power) Law. It now follows that for

$j = 1, \ldots, t,$

$$\Sigma_j = \begin{bmatrix} \beta_j^2 \, \mathrm{Var}(v_j) + \mathrm{Var}(w_j) & \beta_j \, \mathrm{Var}(v_j) \\ \beta_j \, \mathrm{Var}(v_j) & \mathrm{Var}(v_j) \end{bmatrix}$$

To use the above scheme as a system equation of the Kalman-Filter, we must specify $\mathrm{Var}(v_j)$, $\mathrm{Var}(w_j)$ and the starting values θ_o and ϕ_o. The quantity $\mathrm{Var}(w_j)$ reflects our faith in the constancy of the prescribed Law (Arrhenius or Power) in going from one stress level to another, whereas the quantity $\mathrm{Var}(w_j)$ specifies our faith in the validity of the prescribed Law. Once the above have been done, an implementation of the Kalman-Filter algorithm is straightforward.

Strategy 2. For purposes of discussion, we focus attention on the j-th stress level, and suppose that $T_{j1} < T_{j2} < \ldots < T_{jn}$ are the ordered times of failure of the n items tested under S_j; that is T_{jk}, $k = 1, \ldots, n$ are the order statistics of T_j. Let t_{jk} be the realization of T_{jk}. It is well known that $\bar{G}_j(u)$, the survival function of T_j, is given by $\bar{G}_j(u) = \exp(-\int_o^u h_j(u)du)$, where $h_j(\cdot)$ is the failure rate of T_j. Let $H_j(u) = \int_o^u h_j(s)ds$, then under the Power Law $\theta_j = \alpha_j S_j^{\phi_j}$, $\bar{G}_j(T_{jk}) = \exp[-H(T_{jk})] = \exp[-\alpha_j S_j^{\phi_j} T_{jk}]$. The above motivates us to define $y_{jk}^* = \log t_{jk}$, $F_j = (-1, -\log S_j)$, $\alpha_{jk} = (\log \alpha_j, \phi_j)$, $v_{jk}^* = \log\{-\log \bar{G}_j(T_{jk})\}$, and consider, as a model of observations $y_{jk} = F_j \alpha_{jk} + v_{jk}$, where $y_{jk} = y_{jk}^* - E(v_{jk}^*)$ and $v_{jk} = v_{jk}^* - E(v_{jk}^*)$. Since the distribution of v_{jk}^* is known — note that $\bar{G}_j(t_{jk})$ has a beta distribution — the quantity $E(v_{jk}^*)$ is known. Even though we know the distribution of v_{jk}, we are unable to analytically assert that it is Gaussian. However, some empirical work suggests that the assumption of Gaussianity for v_{jk} is reasonable when: n is as small as 10 and k close to n; n is

greater than 10 and k moderate to large; n is greater than 25 and k as small as 6. It should also be clear that the v_{jk}'s are correlated for k = 1, ..., n, and that the correlations are known; this information should be incorporated in the filtering algorithm.

For the system equation, we propose the steady-model of Harrison and Stevens (1976); that is

$$\alpha_{jk} = \begin{cases} G_j \alpha_{j(k-1)}, & \text{when the last observed failure is } y_{j(k-1)}, \\ G_j \alpha_{(j-1)k} + w_{jk}, & \text{when the last observed failure is } y_{(j-1)k}; \end{cases}$$

G_j is a 2x2 identity matrix and $w_{jk} = (w_{1,jk}, w_{2,jk})$. The innovations $w_{1,jk}$ and $w_{2,jk}$ are assumed to be independent and Gaussian with mean 0 and covariance Σ_w, assumed known. The entries for Σ_w reflect our faith about the constancy of α_j and ϕ_j in going from one stress level to the other. A computer program which implements the filtering algorithm under the above scheme has been used by us on data from a realistic scenario and also some simulated data; the details can be found in Blackwell and Singpurwalla (1988).

3. A KALMAN-FILTER MODEL FOR DOSE-RESPONSE EXPERIMENTS

Following the notaiton of Section 1, we let $Y^*(x) = \log\{-\log Y(x)\}$ and require that $Y^*(x) \sim \mathcal{N}(\mu(x), \sigma^2(x))$; this implies that Y(x) must have a double-lognormal distribution with parameters $\mu(x)$ and $\sigma^2(x)$. The density function of this distribution is flexible enough to represent a variety of subjective opinions on Y(x); when $\sigma^2(x)$ is small the mean of Y(x) is $\exp(-\exp(\mu(x))$, and so is its median. However, since we have assumed that the relationship between the dose and the

response is of the form $E(Y(x)) = \exp(-\alpha(x)x^{\beta(x)})$, it follows that $\mu(x) = \log\alpha(x) + \beta(x)\log x = E(Y^*(x))$. The above therefore prompts us to write, as a model of observations the relationship

$$Y^*(x_j) = [1, \ \log x_j] \begin{bmatrix} \gamma \\ \beta \end{bmatrix}_j + r_j ,$$

where $(\gamma, \beta)'_j = (\gamma_j, \beta_j)'$ with $\gamma_j = \log(x_j)$, $\beta_j = \beta(x_j)$, and $r_j \sim N(0, \sigma^2(x_j))$.

For the system equation of the Kalman-Filter, we propose the steady model

$$(\gamma, \beta)'_j = (\gamma, \beta)'_{j-1} + w_j$$

with $w_j \sim N(0, W_j)$. The quantities $\sigma^2(x_j)$ and W_j have to be specified by us; some hints of choosing these, and other related matters are given in Meinhold and Singpurwalla (1987). Once the above have been done, an application of the Kalman-Filter mechanism proceeds routinely.

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